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1991 J. Phys. A: Math. Gen. 24 1741

(http://iopscience.iop.org/0305-4470/24/8/017)

J. Phys. A: Math. Gen. 24 (1991) 1741–1750. Printed in the UK

Iterated scattering map for rearrangement scattering

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Received 4 June 1990, in final form 14 January 1991

Abstract. An iterated scattering map for three-particle-rearrangement scattering is constructed by forming sequences of scattering trajectories. This map works in a similar way to the well known Poincaré map for bound trajectories. It provides useful information for investigating the integrability properties of the system. A numerical example is shown for the Calogero–Moser system with hyperbolic potential functions.

1. Introduction

In two previous papers [1, 2] an iterated scattering map M was constructed for elastic and inelastic two-particle scattering. In a first step, a map from the set of incoming asymptotes into the set of outgoing asymptotes is given by following the actual scattering trajectories of the system. In a second step incoming and outgoing asymptotes with identical values for the impact parameter and momentum are identified. In inelastic scattering the values of the internal variables are also kept the same. The composition of these two steps provides a map M from the set of incoming asymptotes into itself. This map can be iterated. In the case of a chaotic deflection function, this map is not well defined on a subset of measure zero (the set of pre-images of the stable manifolds of unstable localized orbits). This is not a serious problem for the numerical construction of the map.

The main motivation of this construction was to obtain a numerical test for the existence of a second conserved quantity K, which fits together with the asymptotic properties of the Hamiltonian H [3]. By fitting together we mean: $\{K, H\} = 0$ and $\{K_0, H_0\} = 0$, where K_0 and H_0 are the asymptotic limit forms of K and H. This kind of asymptotically compatible integrability will be explained in section 3 in more detail. If the set of all asymptotes is foliated into lower dimensional subsets invariant under M, then there exists a conserved quantity, which is independent of H. If a second conserved quantity does not exist at all or if the asymptotic properties of the second conserved quantity do not fit together with the asymptotic properties of H, then the iterated map M shows chaotic behaviour of the kind which is typical for a perturbed twist map.

Recently the scattering map M turned out to be important for the investigation of quantum scattering chaos [4]. The unstable periodic points of M determine the statistical properties of the semiclassical S-matrix and they determine whether the S-matrix behaves like the one for a random matrix system. A behaviour of random matrix type is believed to be the quantum criterion for chaos.

This remarkable observation leads to an intriguing problem: for the quantum S-matrix to possess generic properties it is sufficient for the iterated classical scattering map to be chaotic. It is not necessary for the classical deflection function to be chaotic, which has been taken as the criterion for classical scattering chaos (see e.g. the review in [5]). However, if the deflection function is chaotic, then M is even more chaotic. However, M can be chaotic without a chaotic deflection function. With these relationships in mind, we have to think seriously about the idea of taking the properties of the iterated scattering map as the criteria for scattering integrability of classical systems.

We have mentioned these problems in order to give some additional motivation for considering the iterated scattering map at all. In this paper, however, we will not investigate the connection between M and the quantum behaviour; instead we will elaborate on the construction of M itself. Because of the open problems, in which M is involved, it is desirable to gain better insight into the properties of M for many different systems and preferably for systems of different qualitative structure. So far the construction of M has only been given for two-particle scattering without rearrangement. However, because of the new interest in M we feel justified in generalizing its construction to reactive scattering with rearrangement in this paper. In section 2 the construction of M is given. In section 3 we show the way in which M provides information on the integrability properties of the system and in section 4 we illustrate the ideas presented by the numerical example of the Calogero–Moser system with hyperbolic potential functions. Section 5 contains some final remarks.

2. Construction of the scattering map

The map to be constructed operates on the set of all incoming asymptotes, which will be denoted by $\mathcal{A}^{\mathrm{in}}$ in the following. First we label the asymptotes in an appropriate way.

For the moment let us assume that the system consists of three point particles A, B and C without any internal degrees of freedom. The generalization to more than three particles or to particles with additional internal degrees of freedom is straightforward; but it only makes the notation more cumbersome.

Let the Hamiltonian of the complete system be

$$H = \frac{p_{\mathrm{A}}^2}{2m_{\mathrm{A}}} + \frac{p_{\mathrm{B}}^2}{2m_{\mathrm{B}}} + \frac{p_{\mathrm{C}}^2}{2m_{\mathrm{C}}} + V_{\mathrm{A}}(q_{\mathrm{B}} - q_{\mathrm{C}}) + V_{\mathrm{B}}(q_{\mathrm{C}} - q_{\mathrm{A}}) + V_{\mathrm{C}}(q_{\mathrm{A}} - q_{\mathrm{B}}). \tag{1}$$

The position coordinate of particle i is $\boldsymbol{q}_i$, its momentum is $\boldsymbol{p}_i$ and its mass is m_i. The two-particle potentials V_i should be sufficiently fast decreasing, so that all the asymptotic conditions of scattering theory are fulfilled.

We have four different asymptotic arrangement channels. In channel 0 all three particles are free and well separated from each other. The asymptotic Hamiltonian of channel 0 is just the kinetic energy

$$H_0 = \frac{p_{\mathrm{A}}^2}{2m_{\mathrm{A}}} + \frac{p_{\mathrm{B}}^2}{2m_{\mathrm{B}}} + \frac{p_{\mathrm{C}}^2}{2m_{\mathrm{C}}}. \tag{2}$$

In channel A particle A is free and well separated from B and C, which form a bound state. The asymptotic Hamiltonian in channel A is

$$H_{\mathrm{A}} = H_0 + V_{\mathrm{A}}(q_{\mathrm{B}} - q_{\mathrm{C}}). \tag{3}$$

Channels B and C and their asymptotic channel Hamiltonians are defined in an analoguous way by interchanging the indices A, B and C.

The system has nine degrees of freedom and the phase space is eighteen-dimensional. The motion of the centre of mass can be separated out and is described by giving the values of the total momentum

$$\boldsymbol{P}_S = \boldsymbol{p}_{\mathrm{A}} + \boldsymbol{p}_{\mathrm{B}} + \boldsymbol{p}_{\mathrm{C}} \tag{4}$$

and the position of the centre of mass

$$\boldsymbol{S} = (m_{\mathrm{A}}\boldsymbol{q}_{\mathrm{A}} + m_{\mathrm{B}}\boldsymbol{q}_{\mathrm{B}} + m_{\mathrm{C}}\boldsymbol{q}_{\mathrm{C}})/M \tag{5}$$

where $M = m_{\mathrm{A}} + m_{\mathrm{B}} + m_{\mathrm{C}}$. For simplicity we describe all scattering events in the centre of mass coordinate system, where $\boldsymbol{S} \equiv 0 \equiv \boldsymbol{P}_S$.

We label the asymptotes by giving eleven quantities for the relative motion, which are conserved in the asymptotic region. For the asymptotes in channel A we observe the motion of a free particle A against a bound state formed of the particles B and C. For the description of this asymptotic motion we use the coordinate

$$\boldsymbol{Q}_{\mathrm{A}} = \frac{m_{\mathrm{C}}\boldsymbol{q}_{\mathrm{C}} + m_{\mathrm{B}}\boldsymbol{q}_{\mathrm{B}}}{m_{\mathrm{C}} + m_{\mathrm{B}}} - \boldsymbol{q}_{\mathrm{A}} \tag{6}$$

and its conjugate momentum

$$\boldsymbol{\Pi}_{\mathrm{A}} = [m_{\mathrm{A}}(\boldsymbol{p}_{\mathrm{B}} + \boldsymbol{p}_{\mathrm{C}}) - (m_{\mathrm{B}} + m_{\mathrm{C}})\boldsymbol{p}_{\mathrm{A}}]/M. \tag{7}$$

The corresponding reduced mass is

$$\mu_{\mathrm{A}} = m_{\mathrm{A}}(m_{\mathrm{B}} + m_{\mathrm{C}})/M. \tag{8}$$

For the internal motion of the B–C bound state we use the coordinate

$$\boldsymbol{y}_{\mathrm{A}} = \boldsymbol{q}_{\mathrm{B}} - \boldsymbol{q}_{\mathrm{C}} \tag{9}$$

and its conjugate momentum

$$\boldsymbol{U}_{\mathrm{A}} = \frac{m_{\mathrm{C}}\boldsymbol{p}_{\mathrm{B}} - m_{\mathrm{B}}\boldsymbol{p}_{\mathrm{C}}}{m_{\mathrm{B}} + m_{\mathrm{C}}}. \tag{10}$$

The corresponding reduced mass is

$$\sigma_{\mathrm{A}} = m_{\mathrm{B}}m_{\mathrm{C}}/(m_{\mathrm{B}} + m_{\mathrm{C}}). \tag{11}$$

In these coordinates the channel Hamiltonian H_{A} has the form

$$H_{\mathrm{A}} = \frac{\boldsymbol{P}_S^2}{2M} + \frac{\boldsymbol{\Pi}_{\mathrm{A}}^2}{2\mu_{\mathrm{A}}} + \frac{\boldsymbol{U}_{\mathrm{A}}^2}{2\sigma_{\mathrm{A}}} + V_{\mathrm{A}}(\boldsymbol{y}_{\mathrm{A}}). \tag{12}$$

The relative free motion between the two fragments is determined by treating $\boldsymbol{\Pi}_{\mathrm{A}}$ and a corresponding two-component impact parameter $\boldsymbol{b}_{\mathrm{A}}$ in the same way as in [1] for the scattering of two structureless particles. For the internal motion of the B–C bound state let us introduce the internal Hamiltonian

$$h_{\mathrm{A}} = \frac{U_{\mathrm{A}}^2}{2\sigma_{\mathrm{A}}} + V_{\mathrm{A}}(y_{\mathrm{A}}). \tag{13}$$

If h_{A} is not completely integrable and the internal motion of the B–C system can be chaotic, then the scattering map to be constructed can also only be chaotic. More interesting are those cases in which all internal motions of two-particle fragments are completely integrable. From now on we assume this property to hold for our system under investigation. Then h_{A} can be written as a function of a three-component internal action variable $\boldsymbol{I}_{\mathrm{A}}$ only and it does not depend on the conjugate angle variable $\boldsymbol{\varphi}_{\mathrm{A}}$.

$\boldsymbol{I}_{\mathrm{A}}$ is constant in the asymptotic region of channel A and can be used to label asymptotes. $\boldsymbol{\varphi}_{\mathrm{A}}$ moves asymptotically according to the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\varphi}_{\mathrm{A}} = \boldsymbol{\omega}_{\mathrm{A}} = \frac{\partial h_{\mathrm{A}}}{\partial \boldsymbol{I}}. \tag{14}$$

In analogy to what has been done for inelastic scattering in [2] we introduce a reduced angle $\boldsymbol{\psi}_{\mathrm{A}}$ by subtracting the asymptotic motion from the actual $\boldsymbol{\varphi}_{\mathrm{A}}$. $\boldsymbol{\psi}_{\mathrm{A}}$ is constant along asymptotes of channel A and it gives the relative phase between the internal and external motion.

$$\boldsymbol{\psi}_{\mathrm{A}} = \boldsymbol{\varphi}_{\mathrm{A}} - \boldsymbol{\omega}_{\mathrm{A}}\mu_{\mathrm{A}}\boldsymbol{\Pi}_{\mathrm{A}} \cdot \boldsymbol{Q}_{\mathrm{A}}/\boldsymbol{\Pi}_{\mathrm{A}}^2. \tag{15}$$

Such reduced phases have been used before to describe asymptotes of molecular scattering motion [6, 7].

It may be appropriate to use the total energy E as one of the asymptotic variables because of the conservation of energy. We can replace $\boldsymbol{\Pi}_{\mathrm{A}}$ by E and by $\boldsymbol{\alpha}_{\mathrm{A}}$, the two-component direction of $\boldsymbol{\Pi}_{\mathrm{A}}$. In total we label any incoming asymptote of channel A by the values of: $E, \boldsymbol{\alpha}_{\mathrm{A}}, \boldsymbol{b}_{\mathrm{A}}, \boldsymbol{I}_{\mathrm{A}}, \boldsymbol{\psi}_{\mathrm{A}}$.

The asymptotic variables of channels B and C are obtained in the same way by an interchange of indices in equations (6)–(15).

In channel 0 all momenta are conserved. Therefore we may use the following quantities to label asymptotes in channel 0: Relative momentum $\boldsymbol{U}_{\mathrm{A}}$ between particles B and C as defined in (10); relative momentum $\boldsymbol{U}_{\mathrm{C}}$ between particles A and B defined in analogy to (10) by interchanging indices. Instead of one of these momenta—e.g. $\boldsymbol{U}_{\mathrm{C}}$—we can also use the total kinetic energy E and the direction $\boldsymbol{\gamma}_{\mathrm{C}}$ of $\boldsymbol{U}_{\mathrm{C}}$. In addition we use the impact parameter $\boldsymbol{b}_{\mathrm{C}}$ of particle A relative to particle B and the impact parameter $\boldsymbol{b}_{\mathrm{A}}$ of particle C relative to particle B. As a last quantity we need a relative shift between the A–B motion and the B–C motion. We may use

$$\psi_0 = \sigma_{\mathrm{A}}\boldsymbol{y}_{\mathrm{A}} \cdot \boldsymbol{U}_{\mathrm{A}}/\boldsymbol{U}_{\mathrm{A}}^2 - \sigma_{\mathrm{C}}\boldsymbol{y}_{\mathrm{C}} \cdot \boldsymbol{U}_{\mathrm{C}}/\boldsymbol{U}_{\mathrm{C}}^2. \tag{16}$$

Direct computation shows that $(\mathrm{d}/\mathrm{d}t)\psi_0 = 0$ under the motion generated by H_0. In total, asymptotes in channel 0 can be labelled by the values of: $E, \boldsymbol{\gamma}_{\mathrm{C}}, \boldsymbol{U}_{\mathrm{A}}, \boldsymbol{b}_{\mathrm{A}}, \boldsymbol{b}_{\mathrm{C}}, \psi_0$. Outgoing asymptotes can be labelled in a similar way to the incoming ones.

By Φ we denote the flow generated by H, and by Φ_n we denote the flow generated by H_n.

Compared with two-particle scattering the decomposition of $\mathcal{A}^{\text{in}}$ is new. In it the complete set of all asymptotes is decomposed into several connected components, one for each arrangement channel

$$\mathcal{A}^{\text{in}} = \mathcal{A}_0^{\text{in}} \cup \mathcal{A}_{\text{A}}^{\text{in}} \cup \mathcal{A}_{\text{B}}^{\text{in}} \cup \mathcal{A}_{\text{C}}^{\text{in}}.$$

We construct the map $M : \mathcal{A}^{\text{in}} \to \mathcal{A}^{\text{in}}$ using the same basic procedure as in [1, 2]: First we fix some value E of the total energy, which will be conserved all along the iteration process. Next we choose an arbitrary initial point $x_0 \in \mathcal{A}_{n_0}^{\text{in}}$, which belongs to the correct energy E and where n_0 labels any energetically open channel. We construct the scattering trajectory with initial condition x_0. This trajectory is created by applying Φ on the initial condition x_0.

If x_0 does not belong to the exceptional subset of measure zero of trajectories which will get stuck in the interaction region and which does not possess an outgoing asymptote, then the trajectory finally reaches the outgoing asymptotic region at some final point $x_0' \in \mathcal{A}_{n_0'}^{\text{out}}$. Next we switch to the asymptotic Hamiltonian $H_{n_0'}$ as generator of the dynamics, let the time run backwards and follow the trajectory starting in x_0' under $\Phi_{n_0'}^{-1}$, i.e. the time-reversed motion generated by $H_{n_0'}$. Under this motion all the quantities $(n_0', E, \boldsymbol{\alpha}_{n_0'}, \boldsymbol{b}_{n_0'}, \boldsymbol{I}_{n_0'}, \psi_{n_0'})$ in the case of $n_0' \in \{\text{A}, \text{B}, \text{C}\}$ or $(E, \gamma_{\text{C}}, \boldsymbol{U}_{\text{A}}, \boldsymbol{b}_{\text{A}}, \boldsymbol{b}_{\text{C}}, \psi_0)$ in the case of $n_0' = 0$ are constant. Finally this time-reversed trajectory arrives at some initial asymptote $x_1 \in \mathcal{A}_{n_1}^{\text{in}}$ in channel n_1, where $n_1 = n_0'$. The numerical values of all components of x_1 coincide with those of x_0'. Thereby the map

$$\begin{aligned} M :& \mathcal{A}^{\text{in}} \to \mathcal{A}^{\text{in}} \\ & x_0 \mapsto x_1 \end{aligned}$$

is completed and the next step of the iteration can start at the point x_1.

The combination of the motion generated by the full Hamiltonian with the time-reversed motion generated by the asymptotic Hamiltonian parallels the construction of the quantum S-operator, where the evolution operator for the full Hamiltonian is combined with the inverse evolution operator for the asymptotic Hamiltonian. This parallelity between M and S explains the connections between classical and quantum properties found in [4].

3. Integrability properties

Let us assume that a function K, which is independent of H, exists on the phase space, such that

$$\{K, H\} = 0. \tag{17}$$

In addition let conserved channel quantities $K_n, n \in \{0, \text{A}, \text{B}, \text{C}\}$ exist, such that

$$\{K_n, H_n\} = 0 \tag{18}$$

for all $n \in \{0, \text{A}, \text{B}, \text{C}\}$ and K_n is the asymptotic limit of K in channel n. For example

$$K_{\text{A}} = \lim_{\text{A}} K \tag{19}$$

where the limit is to be taken such that $| q_B - q_A | \to \infty$, $| q_C - q_A | \to \infty$ and $| q_B - q_C |$ stays small. We take corresponding limits to define the other K_n. K is conserved under Φ. In the outgoing asymptotic region in channel n the values of K and K_n coincide. Φ_n^{-1} conserves the value of K_n and in the incoming asymptotic region, where we switch back to Φ, the values of K_n and K coincide again. Therefore, the value of K is conserved under the complete action of the map M and the asymptotic space $\mathcal{A}^{\text{in}}$ is foliated into level sets of K, which are invariant under M. Let us call a conserved quantity with these properties an 'asymptotically compatible constant of motion'.

Let the system have N degrees of freedom and accordingly the asymptotic space $\mathcal{A}^{\text{in}}$ for a fixed value of E is $2(N-1)$-dimensional. Let us assume that the system fulfils the following conditions.

There exist $N-1$ independent asymptotically compatible constants of motion $K^{(j)}, j = 1, \ldots, N-1$ in involution, i.e. $\{K^{(j)}, K^{(i)}\} = 0$ and $\{K^{(j)}, H\} = 0$ for all i, j. The asymptotic limit of $K^{(j)}$ in channel n is $K_n^{(j)}$ with $\{K_n^{(j)}, K_n^{(i)}\} = 0$ and $\{K_n^{(j)}, H_n\} = 0$ for all $i, j = 1, \ldots, N-1$ and all n. In such a case the system is completely integrable in the sense of Liouville and, in addition, this integrability is transferred to the asymptotic spaces $\mathcal{A}_n$, which are foliated into $(N-1)$-dimensional level sets of the functions $K_n^{(j)}$. This foliation is conserved under the iterated map M. Let us call such a situation 'complete asymptotically compatible integrability'.

Now the question arises as to whether we can turn the argument around and whether the system has the property of complete asymptotically compatible integrability whenever there is a smooth foliation of $\mathcal{A}$ into $(N-1)$-dimensional level sets and this foliation is conserved under M. Unfortunately, there is no simple answer. As a preliminary step we consider a system with two essential degrees of freedom and a two-dimensional asymptotic space $\mathcal{A}$ for each value of E like the example in section 4. Assume that we apply the iterated scattering map and find a foliation of $\mathcal{A}$ into one-dimensional subsets, which are invariant under M. In a first step we construct a function J on $\mathcal{A}^{\text{in}}$ such that the level sets of J are the subsets invariant under M. In a second step we construct a continuation of J on the total phase space. To each point p of the phase space we take the trajectory through p under Φ and follow it backwards into the incoming asymptotic region and arrive at some point $x(p) \in \mathcal{A}^{\text{in}}$ and define

$$K(p) = J(x(p)) \tag{20}$$

per construction K is invariant under Φ. The existence of the foliation of $\mathcal{A}^{\text{in}}$ which is invariant under M together with the assumed complete integrability of the channel Hamiltonians H_n excludes the possibility that the flow Φ is chaotic. The time-reversed flow Φ_n^{-1} could not combine with a chaotic Φ to give an integrable scattering map M in total. As long as Φ is not chaotic, the construction of K previously outlined gives a non-chaotic function K. The problem is to demonstrate the smoothness and differentiability properties of K constructed in this way. Even if H is a smooth function, it is not automatically guaranteed that the K created by the transport of asymptotic initial conditions is also smooth (for some discussion of related problems see [8]).

If for a system with N degrees of freedom the iterated scattering map provides a foliation of $\mathcal{A}$ for fixed E into $(N-1)$-dimensional subsets, then we construct in a first step $(N-1)$ quantities $J^{(j)}$ defined on $\mathcal{A}$, which are independent and in involution and whose level sets generate the foliation created by M. In a second step we transport them by the flow in order to obtain corresponding quantities $K^{(j)}$ defined on the whole phase space. However, we again run into the same smoothness problems as in the case of two degrees of freedom. In the case of the smoothness of all $K^{(j)}$ complete asymptotically compatible integrability would be established.

Because of the existence of the foliation of $\mathcal{A}^{\text{in}}$ under M the values of $K^{(j)}$ must be conserved under the feedback of outgoing asymptotes into incoming asymptotes by Φ_n^{-1} i.e. the asymptotic limit forms of $K^{(j)}$ must be a set of functions $K_n^{(j)}$ each conserved under H_n.

4. Example of the Calogero–Moser system

The ideas presented so far will be illustrated by an example for a particular Hamiltonian. To make things as simple as possible, we take a one-dimensional position space, so that after the separation of the centre of mass motion only two essential degrees of freedom remain. $\mathcal{A}^{\text{in}}$ is three-dimensional. Because of the conservation of the total energy E we only need two-dimensional subsets of $\mathcal{A}^{\text{in}}$ belonging to a fixed value of E and plots of the iterated scattering map M can be presented. For the case of a one-dimensional position space we use the following coordinates in $\mathcal{A}^{\text{in}}$: E and (I_n, ψ_n) for the channels A, B, C or (U_{A}, ψ_0) for channel 0.

A good example for demonstration is the Calogero–Moser system with hyperbolic potential functions [9]:

$$H = \frac{p_{\text{A}}^2}{2} + \frac{p_{\text{B}}^2}{2} + \frac{p_{\text{C}}^2}{2} - D_{\text{A}}[\cosh(q_{\text{B}} - q_{\text{C}})]^{-2} + D_{\text{B}}[\sinh(q_{\text{A}} - q_{\text{C}})]^{-2} - D_{\text{C}}[\cosh(q_{\text{A}} - q_{\text{B}})]^{-2}. \tag{21}$$

We have set $m_{\text{A}} = m_{\text{B}} = m_{\text{C}} = 1$ leading to $M = 3, \mu_n = \frac{2}{3}, \sigma_n = \frac{1}{2}$ for $n \in \{\text{A}, \text{B}, \text{C}\}$. The potential strengths D_n are free parameters. In terms of the variables defined in (6)–(15) we write H in the form

$$\begin{aligned} H = \frac{P_S^2}{6} + \frac{3\Pi_{\text{A}}^2}{4} + U_{\text{A}}^2 - D_{\text{A}}[\cosh(y_{\text{A}})]^{-2} + D_{\text{B}}[\sinh(Q_{\text{A}} + y_{\text{A}}/2)]^{-2} \\ - D_{\text{C}}[\cosh(Q_{\text{A}} - y_{\text{A}}/2)]^{-2}. \end{aligned} \tag{22}$$

The asymptotic limit of channel A is the limit $| Q_{\text{A}} | \to \infty$. Therefore,

$$H_{\text{A}} = \frac{P_S^2}{6} + \frac{3\Pi_{\text{A}}^2}{4} + U_{\text{A}}^2 - D_{\text{A}}[\cosh(y_{\text{A}})]^{-2}. \tag{23}$$

In channel A we use the action-angle variables

$$I_{\text{A}} = \sqrt{D_{\text{A}}} - \sqrt{D_{\text{A}}[\cosh(y_{\text{A}})]^{-2} - U_{\text{A}}^2} \tag{24}$$

$$\varphi_{\text{A}} = \frac{\pi}{4} + \frac{1}{2}\sin^{-1}[(g_{\text{A}} - 2U_{\text{A}}^2[\cosh(y_{\text{A}})]^2)/g_{\text{A}}] \tag{25}$$

where $g_{\text{A}} = D_{\text{A}} + y_{\text{A}}^2 - D_{\text{A}}[\cosh(y_{\text{A}})]^{-2}$. Equations (23)–(25) give

$$h_{\text{A}} = -(\sqrt{D_{\text{A}}} - I_{\text{A}})^2 \tag{26}$$

$$\omega_{\text{A}} = 2(\sqrt{D_{\text{A}}} - I_{\text{A}}). \tag{27}$$

The corresponding expressions for channel C are obtained by an interchange of indices. Channel B does not exist in system (21) because the interaction between A and C is always repulsive and no bound state of these two particles exists.

As has been explained in [9], in the case of $D_{\mathrm{A}} = D_{\mathrm{B}} = D_{\mathrm{C}} = D$ there exists the conserved quantity

$$K = p_{\mathrm{A}}p_{\mathrm{B}}p_{\mathrm{C}} + p_{\mathrm{A}}D[\cosh(q_{\mathrm{B}} - q_{\mathrm{C}})]^{-2} - p_{\mathrm{B}}D[\sinh(q_{\mathrm{A}} - q_{\mathrm{C}})]^{-2} + p_{\mathrm{C}}D[\cosh(q_{\mathrm{A}} - q_{\mathrm{B}})]^{-2}. \tag{28}$$

In this integrable case the trajectories always end in channel C, when they start in channel A and *vice versa*. In addition, the final action I_{C} has the same value as the initial action I_{A}. The only effect of the scattering process is a shift of ψ, whose value depends on I.

In the limit $| Q_{\mathrm{A}} | \to \infty$, i.e. in the asymptotic limit of channel A we find the following limiting form of K

$$K_{\mathrm{A}} = p_{\mathrm{A}}p_{\mathrm{B}}p_{\mathrm{C}} + p_{\mathrm{A}}D[\cosh(q_{\mathrm{B}} - q_{\mathrm{C}})]^{-2}. \tag{29}$$

A direct computation shows that $\{K_{\mathrm{A}}, H_{\mathrm{A}}\} = 0$. By interchanging indices in (29) we obtain K_{C} fulfilling $\{K_{\mathrm{C}}, H_{\mathrm{C}}\} = 0$. Of course, the numerical values of K and K_n coincide in the limit of channel n. K is conserved under Φ and K_n is conserved under Φ_n^{-1}. The iterated scattering process always stays on the subset of $\mathcal{A}^{\mathrm{in}}$ belonging to one particular value of K, i.e. M is asymptotically compatible integrable and foliates $\mathcal{A}^{\mathrm{in}}$. The leaves are the lines $I =$ constant ranging over both components $\mathcal{A}_{\mathrm{A}}^{\mathrm{in}}$ and $\mathcal{A}_{\mathrm{C}}^{\mathrm{in}}$. M is a pure twist map in this integrable case.

As soon as all three D_n are no longer equal, the integrability of H is destroyed and M no longer foliates $\mathcal{A}^{\mathrm{in}}$ into invariant lines as figures 1 and 2 demonstrate. In figure 1 we have chosen $E = -0.1, D_{\mathrm{A}} = 1, D_{\mathrm{B}} = 1$ and $D_{\mathrm{C}} = 0.8$. Because $E < 0$, channel 0 is closed energetically and $\mathcal{A}^{\mathrm{in}}$ consists of the two connected components $\mathcal{A}_{\mathrm{A}}^{\mathrm{in}}$ and $\mathcal{A}_{\mathrm{C}}^{\mathrm{in}}$. As figure 1 shows, the invariant lines partly still exist and partly they are destroyed and replaced by chaotic regions and secondary structures. Several initial conditions are marked by crosses. Similar structures in the two channels at similar values of I belong to the same initial point. The large chaotic region ranging over both channels is represented by 1000 iterates of the point $I = 0.069, \psi = 3.066$ in channel C.

In figure 2 another example is given for $E = 0.2, D_{\mathrm{A}} = 1, D_{\mathrm{B}} = 1.05$ and $D_{\mathrm{C}} = 0.95$. This time $E > 0$ and channel 0 is open energetically. However, it is not interesting to follow the motion in channel 0. As soon as a sequence of trajectories starting in channel A or C has reached channel 0, it stays in channel 0 in almost all cases and ψ_0 monotonically drifts to large values. In the same way it is not interesting to start the iteration in channel 0 and to follow iterated trajectories before they reach channel A or C. In the plot we do not show $\mathcal{A}_0^{\mathrm{in}}$ at all. If the iteration reaches a point of channel A or C, whose image lies in channel 0, then this point of channel A or C is marked by a small square in the plot.

In figures 1 and 2 we see structures which are typical for perturbed twist maps [10, 11]. Along KAM lines the map always jumps from channel A to channel C and *vice versa* in the integrable case. In the chaotic region we find points whose image lies in the same channel. In our figures this mainly happens in channel A, because for our parameter values the binding force between B and C is stronger than the binding force between A and B.

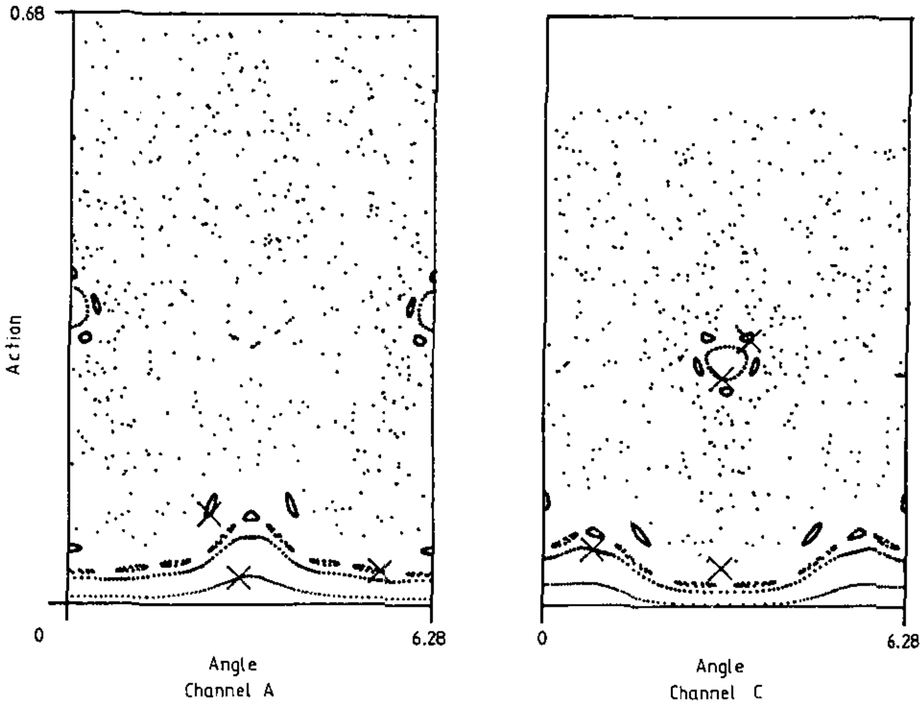

Figure 1. Iterated scattering map for system (21) for parameter values $D_A = 1, D_B = 1, D_C = 0.8$ and energy $E = -0.1$. A few hundred iterates of some initial points (marked by crosses) are plotted.

One remark on the frame boundaries in the figures: the maximally possible value for I_n in channel n is given by

$$I_{n,\max} = \sqrt{D_n} - \sqrt{-E} \qquad \text{if } E < 0$$
$$I_{n,\max} = \sqrt{D_n} \qquad \text{if } E > 0.$$

The figures show that if the D_n are not equal, then $\mathcal{A}^{\text{in}}$ is not foliated into lines invariant under M and asymptotically compatible integrability does not exist.

5. Final remarks

In this paper the iterated scattering map has been constructed for rearrangement scattering. For simplicity we have considered the case of three structureless point particles. In the case of particles with internal degrees of freedom we add additional components to the quantities I and ψ introduced in section 2, add the corresponding dimensions to $\mathcal{A}$ and proceed as before. Thereby we combine the method of [2] to treat inelastic scattering with internal degrees of freedom with the method to treat rearrangements given in this paper. The case of more than three fragments can be treated following the same basic pattern as in the case of three fragments shown here. We iteratively add the motion of a further particle relative to the centre of mass of the other particles already present before. This creates a hierarchy of relative position and momentum coordinates and reduced masses. Equations (6)–(16) are the first two steps in this hierarchy.

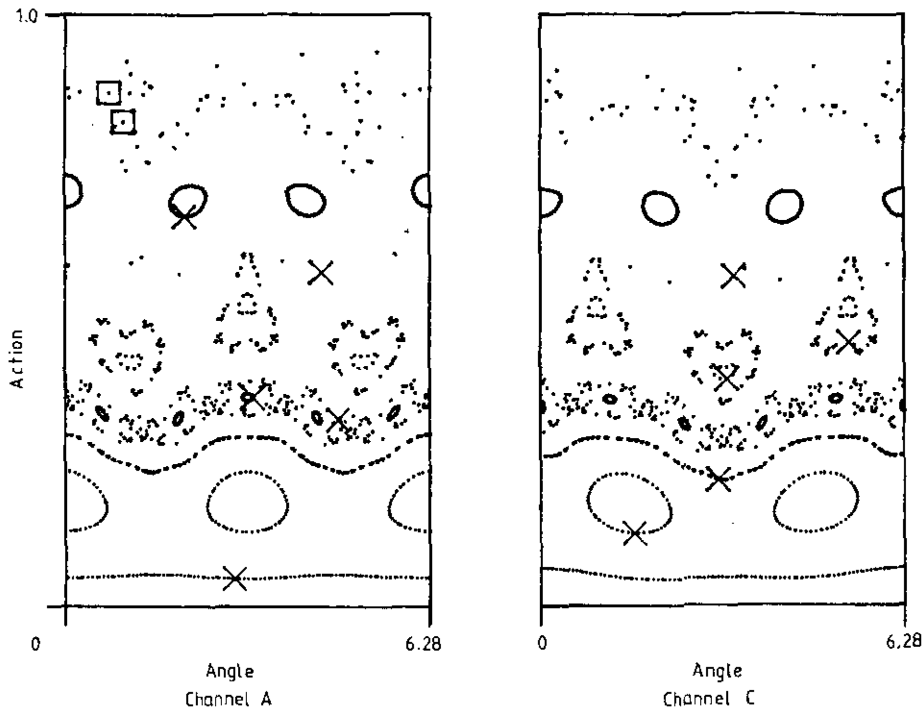

Figure 2. Iterated scattering map for system (21) for parameter values $D_A = 1, D_B = 1.05, D_C = 0.95$ and energy $E = 0.2$. A few hundred iterates of some initial points (marked by crosses) are plotted. Points whose images lie in channel 0 are marked by squares.

The example of section 4 indicates that the complete integrability of the iterated scattering map is an exceptional case which is not structurally stable under perturbations of the system. In accordance, M becomes chaotic in the generic case in the same way as a pure twist map becomes chaotic under generic perturbations.

Unfortunately we could not give a definite answer to the problem of equivalence between complete asymptotically compatible integrability and the absence of chaos in the iterated scattering map. Whereas the conclusion is evident in one direction it remains an open problem to find sufficient conditions for the reverse direction to hold.

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